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DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND HIGHER DEGREE

3.1 INTRODUCTION

So far, we have discussed differential equations of the first order and first degree. Now we shall study differential equations of the first order and degree higher than the first. For convenience, we denote $\frac{dy}{dx}$ by p.

A differential equation of the first order and n^{th} degree is of the form

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots\ldots + P_n = 0 \qquad \ldots(1)$$

where $P_1, P_2, \ldots\ldots, P_n$ are functions of x and y.

Since it is a differential equation of the first order, its general solution will contain only one arbitrary constant.

In the various cases which follow, the problem is reduced to that of solving one or more equations of the first order and first degree.

3.2 EQUATIONS SOLVABLE FOR p

Resolving the left hand side of (1) into n linear factors, we have

$$[p - f_1(x, y)]\,[p - f_2(x, y)], \ldots\ldots, [p - f_n(x, y)] = 0$$

which is equivalent to $p - f_1(x, y) = 0,\ p - f_2(x, y) = 0, \ldots\ldots, p - f_n(x, y) = 0$

Each of these equations is of the first order and first degree and can be solved by the methods already discussed.

If the solutions of the above n component equations are

$$F_1(x, y, c) = 0,\ F_2(x, y, c) = 0, \ldots\ldots, F_n(x, y, c) = 0$$

then the general solution of (1) is given by

$$F_1(x, y, c) \cdot F_2(x, y, c) \ldots\ldots F_n(x, y, c) = 0.$$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve:* $x^2\left(\frac{dy}{dx}\right)^2 + xy\frac{dy}{dx} - 6y^2 = 0.$

Sol. The given equation is $x^2p^2 + xyp - 6y^2 = 0$ where $p = \frac{dy}{dx}$

Factorising $(xp + 3y)(xp - 2y) = 0$

$\Rightarrow$ $xp + 3y = 0$ or $xp - 2y = 0$

Now, $xp + 3y = 0$

$\Rightarrow$ $x\frac{dy}{dx} + 3y = 0$ or $\frac{dy}{y} + 3\frac{dx}{x} = 0$

Integrating, $\log y + 3\log x = \log c$ or $x^3y = c$

Also, $xp - 2y = 0$

$\Rightarrow$ $x\frac{dy}{dx} - 2y = 0$ or $\frac{dy}{y} - 2\frac{dx}{x} = 0$

Integrating, $\log y - 2\log x = \log c$ or $\frac{y}{x^2} = c$ or $y = cx^2$

$\therefore$ The general solution of the given equation is $(x^3y - c)(y - cx^2) = 0$.

Example 2. *Solve* $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0.$

Sol. Solving the given equation for p, we have

$$p = \frac{-(3x^2 - 2y^2) \pm \sqrt{(3x^2 - 2y^2)^2 + 24x^2y^2}}{2xy}$$

$$= \frac{(2y^2 - 3x^2) \pm (3x^2 + 2y^2)}{2xy} = \frac{2y}{x} \text{ or } -\frac{3x}{y}$$

Now, $p = \frac{2y}{x} \Rightarrow \frac{dy}{dx} = \frac{2y}{x}$ or $\frac{dy}{y} - \frac{2dx}{x} = 0$

Integrating, $\log y - 2\log x = \log c$ or $\frac{y}{x^2} = c$ or $y = cx^2$

Also, $p = -\frac{3x}{y} \Rightarrow \frac{dy}{dx} = -\frac{3x}{y}$ or $ydy + 3xdx = 0$

Integrating, $\frac{y^2}{2} + \frac{3x^2}{2} = C$ or $y^2 + 3x^2 = c$

$\therefore$ The general solution of the given equation is $(y - cx^2)(y^2 + 3x^2 - c) = 0$.

Example 3. *Solve* $p^2 + 2py \cot x = y^2.$

Sol. The given equation can be written as $(p + y\cot x)^2 = y^2(1 + \cot^2 x)$

or $p + y\cot x = \pm y \operatorname{cosec} x$

$\therefore$ The component equations are

$$p = y(-\cot x + \operatorname{cosec} x) \quad \ldots(1)$$

and

$$p = y(-\cot x - \operatorname{cosec} x) \quad \ldots(2)$$

From (1), $\frac{dy}{dx} = y\,(-\cot x + \operatorname{cosec} x)$

or $\frac{dy}{y} = (-\cot x + \operatorname{cosec} x)\,dx$

Integrating, $\log y = -\log \sin x + \log \tan \frac{x}{2} + \log c = \log \frac{c \tan \frac{x}{2}}{\sin x}$

or $$y = \frac{c \tan \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} = \frac{c}{2 \cos^2 \frac{x}{2}} = \frac{c}{1 + \cos x}$$

or $y(1 + \cos x) = c$

From (2), $\frac{dy}{dx} = y(-\cot x - \operatorname{cosec} x)$

or $\frac{dy}{y} = (-\cot x - \operatorname{cosec} x)\,dx$

Integrating, $\log y = -\log \sin x - \log \tan \frac{x}{2} + \log c = \log \frac{c}{\sin x \tan \frac{x}{2}}$

or $$y = \frac{c}{2 \sin^2 \frac{x}{2}} = \frac{c}{1 - \cos x} \quad \text{or} \quad y(1 - \cos x) = c$$

$\therefore$ **The general solution of the given equation is**

$$[y(1 + \cos x) - c]\,[y(1 - \cos x) - c] = 0.$$

TEST YOUR KNOWLEDGE

Solve the following equations:

1. $p^2 - 7p + 12 = 0$
2. $xy \left(\frac{dy}{dx}\right)^2 - (x^2 + y^2) \frac{dy}{dx} + xy = 0$
3. $yp^2 + (x - y)p - x = 0$
4. $x^2 \left(\frac{dy}{dx}\right)^2 + 3xy \frac{dy}{dx} + 2y^2 = 0$
5. $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$
6. $p^2 - 2p \sinh x - 1 = 0$
7. $p\,(p + y) = x\,(x + y)$
8. $4y^2p^2 + 2pxy\,(3x + 1) + 3x^3 = 0.$

Answers

1. $(y - 4x - c)\,(y - 3x - c) = c$
2. $(y^2 - x^2 - c)\,(y - cx) = 0$
3. $(y - x - c)\,(x^2 + y^2 - c) = 0$
4. $(xy - c)(x^2y - c) = 0$
5. $(xy - c)(x^2 - y^2 - c) = 0$
6. $(y - e^x - c)\,(y - e^{-x} - c) = 0$
7. $(y - \frac{1}{2} x^2 + c)\,(y + x + ce^{-x} - 1) = 0$
8. $(y^2 + x^3 - c)(y^2 + \frac{1}{2} x^2 - c) = 0.$

Differential Equation of first order but higher degree

Solve the Equation for P

1 $x^2\left(\frac{dy}{dx}\right)^2 + 3xy\frac{dy}{dx} + 2y^2 = 0$

Sol. Put $\frac{dy}{dx} = P$

$x^2P^2 + 3xyP + 2y^2 = 0$

Solving the given equation for P, ~~we~~

$$P = \frac{-3xy \pm \sqrt{9x^2y^2 - 8x^2y^2}}{2x^2}$$

$P = -3xy \pm \sqrt{x^2y^2} / 2x^2$

$P = -3xy \pm xy / 2x^2$

$P = \frac{-3xy + xy}{2x^2}$ or $P = \frac{-3xy - xy}{2x^2}$

$P = \frac{-2xy}{2x^2}$ or $P = \frac{-4xy}{2x^2}$

$$\frac{dy}{dx} = -\frac{y}{x} \quad \text{or} \quad \frac{dy}{dx} = -\frac{2y}{x}$$

$$\frac{dy}{y} = -\frac{dx}{x} \quad \text{or} \quad \frac{dy}{y} = -2\frac{dx}{x}$$

Integrating

$\log y = -\log x + \log c$

$\log y + \log x = \log c$

$\log xy = \log c$

$xy = c$

$\Rightarrow \quad xy - c = 0 \qquad -(1)$

or

$\log y = -2\log x + \log c$

$\log y = -\log x^2 + \log c$

$\log y + \log x^2 = \log c$

$\log y x^2 = c$

$\Rightarrow \quad x^2 y - c = 0 \qquad -(2)$

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From (1) and (2)
general solution of given equation is:

$(xy-c)(x^2y-c)=0$

2. $xy\left(\frac{dy}{dx}\right)^2-(x^2+y^2)\frac{dy}{dx}+xy=0$

Sol. The given equation is Put $P=\frac{dy}{dx}$

$xyP^2-(x^2+y^2)P+xy=0$

Solve for P

$$P=\frac{(x^2+y^2)\pm\sqrt{(x^2+y^2)^2-4x^2y^2}}{2xy}$$

$$P=\frac{(x^2+y^2)\pm\sqrt{x^4+y^4+2x^2y^2-4x^2y^2}}{2xy}$$

$$P=\frac{(x^2+y^2)\pm\sqrt{x^4+y^4-2x^2y^2}}{2xy}$$

$$P=\frac{(x^2+y^2)\pm\sqrt{(x^2-y^2)^2}}{2xy}$$

$$p = \frac{(x^2+y^2) \pm (x^2-y^2)}{2xy}$$

$$p = \frac{x^2+y^2+x^2-y^2}{2xy}$$

or

$$p = \frac{x^2+y^2-(x^2-y^2)}{2xy}$$

$$p = \frac{2x^2}{2xy} \quad \text{or} \quad p = \frac{2y^2}{2xy}$$

$$p = \frac{x}{y} \quad \text{or} \quad \frac{y}{x}$$

Solve $\frac{dy}{dx} = \frac{x}{y}$

$$y\,dy = x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + \frac{c}{2}$$

$$\Rightarrow y^2 - x^2 = c \quad \text{--- (1)}$$

Solve $\frac{dy}{dx} = \frac{y}{x}$

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$$\frac{dy}{y} = \frac{dx}{x}$$

$$\log y = \log x + \log c$$

$$\log y - \log x = \log c$$

$$\frac{y}{x} = c \qquad -(2)$$

from (1) and (2)
General Solution is

$$(y^2 - x^2 - c)(y - cx) = 0$$

3. $$\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$$

Put $\frac{dy}{dx} = p$

$$p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x}$$

$$\frac{p^2 - 1}{p} = \frac{x^2 - y^2}{xy}$$

$$p^2xy - xy = p(x^2 - y^2)$$

$$xyp^2 - p(x^2 - y^2) - xy = 0$$

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Solve for P, we have

$$P = \frac{(x^2-y^2) \pm \sqrt{(x^2-y^2)^2+4x^2y^2}}{2xy}$$

$$P = \frac{(x^2-y^2) \pm \sqrt{x^4+y^4-2x^2y^2+4x^2y^2}}{2xy}$$

$$P = \frac{(x^2-y^2) \pm \sqrt{(x^2+y^2)^2}}{2xy}$$

$$P = \frac{(x^2-y^2) \pm x^2+y^2}{2xy}$$

$$P = \frac{x^2-y^2+x^2+y^2}{2xy} \text{ or } \frac{x^2-y^2-x^2-y^2}{2xy}$$

$$P = \frac{2x^2}{2xy} \text{ or } \frac{-2y^2}{2xy}$$

$$P = \frac{x}{y} \text{ or } -\frac{y}{x}$$

Solve $P = \frac{x}{y}$

$$\frac{dy}{dx} = \frac{x}{y}$$

$$y\,dy = x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + \frac{c}{2}$$

$$y^2 - x^2 = c \quad -(1)$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\frac{dy}{y} = -\frac{dx}{x}$$

$$\log y = -\log x + \log c$$

$$\log y + \log x = \log c$$

$$xy = c \quad = 0 \quad -(2)$$

From (1) and (2), General Solution is

$$(xy - c)(y^2 - x^2 - c) = 0$$